Time and "angular" dependent backgrounds from stationary axisymmetric solutions

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Backgrounds depending on time and on angular variable, namely, polarized and unpolarized $S^1 \times S^2$ Gowdy models, are generated as the sector inside the horizons of the manifold corresponding to axisymmetric solutions. As is known, an analytical continuation of ordinary D-branes, iD-branes allow one to find S-brane solutions. Simple models have been constructed by means of analytic continuation of the Schwarzschild and the Kerr metrics. The possibility of studying the i-Gowdy models obtained here is outlined with an eye toward seeing if they could represent some kind of generalized S-branes depending not only on time but also on an angular variable.

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I. INTRODUCTION

For some time it has been known that the static solution with spherical symmetry, the Schwarzschild black hole solution,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2(\sin^2\theta d\varphi^2 + d\theta^2) \qquad (1)$$

in the region $r > 2M$, becomes a cosmological model as one crosses the horizon. What was the "radial" direction becomes timelike, and the timelike direction becomes spacelike. In this case the corresponding cosmological model is the well-known Kantowski-Sachs model [1].

$$ds^2 = -\frac{1}{\frac{2M}{t} - 1}dt^2 + \left(\frac{2M}{t} - 1\right)dr^2 + t^2(\sin^2\theta d\varphi^2 + d\theta^2), \qquad (2)$$

The singularity at $t = 0$ is the curvature-singularity of Schwarzschild ($r = 0$), where the curvature is infinite, but the singularity at $t = 2M$ is just a lightlike surface where the curvature is regular and we pass from the Kantowski-Sachs region to the Schwarzschild region. We need two copies of each of these regions to describe the complete casual structure of this spacetime.

In string theory one of the important open problems is the correct treatment of time-dependent backgrounds. In [2] S-branes were first introduced. They are objects arising when Dirichlet boundary conditions on open strings are imposed on the time direction. An S-brane is a topological defect [3], all of whose longitudinal dimensions are spacelike, and consequently they exist only for a moment of time. S-branes have been found as explicit time-dependent solutions of Einstein's equations (dilaton antisymmetric tensor fields are also included in some models) [4] in the same way as black hole-like solutions correspond to p-brane solutions. Some of these solutions are actually best thought of as an analytical continuation of ordinary D-branes, or i D-branes, for short [5].

A simple model has been considered in [6], using only the 4D Einstein equations in vacuum. The Schwarzschild-Kantowski-Sachs model has been utilized to nicely define a simple 4D model for an S-brane. By analytic continuation the spherical space ($k = 1$) is transformed into a hyperbolic space ($k = -1$) to obtain the hyperbolic symmetry SO(2, 1), as suggested by the fact that S-branes are kinks in time. Beginning with the Schwarzschild solution, these authors performed the transformation $t \rightarrow ir$, $r \rightarrow it$, $\theta \rightarrow i\theta$, $\varphi \rightarrow i\varphi$ and $M \rightarrow iP$. By these means they were able to obtain a rotated Penrose diagram for the $k = 0, -1$ S-brane solution with well-defined, time-dependent regions.

As mentioned above, what in Ref. [7] was called "horizon methods" of generating cosmological solutions has a long history, though it never seems thought of as a "method". The idea of this method is to reinterpret a part of a known manifold as a cosmological solution of Einstein's equations. For static and stationary axisymmetric solutions which have horizons, as one crosses the horizon what was the radial direction becomes timelike, and the timelike direction becomes spacelike, and the

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model becomes a time (and ''angular'') dependent cosmological model.

In the Kantowski-Sachs model we see for the first time a problem of global topology. If we insist that our model have a closed topology, we can achieve this by compactifying in the r-direction, but if one crosses the horizon once more, the ''Schwarzschild'' portion is closed in the time direction, which would allow closed timelike lines. This problem has been noted in several ''black hole'' cosmology pairs, the most notable example being the Taub-NUT manifold [8]. Various metrics, including that mentioned above, as well as topological problems are discussed in more detail in Ref. [7].

There has been an enormous amount of work on exact solutions of ''black hole'' type axisymmetric metrics, and a large number of axisymmetric solutions with horizons have been found. It is natural to ask what kind of time and angular dependent backgrounds are generated as the portion of these manifolds inside their horizons. Since axisymmetric metrics are characterized by two commuting Killing vectors, one timelike and the other, associated with a symmetry about an axis, spacelike, it is not surprising that the corresponding models are the two Killing vector models studied exhaustively by Gowdy [9,10]. Since these ''black hole'' models are usually assumed to have compact surfaces that have an S^2 topology, and one compactifies the new $t =$ constant surfaces by identification, they will be Gowdy models with an $S^1 \times S^2$ topology.

The paradigm for such models is the Kerr metric, whose associated cosmological model was presented in Ref. [11]. There are several features of this model that are common to many of the axisymmetric solutions. One is that there are two horizons, an outer horizon and an inner one. The time and angular dependent Gowdy model is represented by the region between these two horizons. Inside the inner horizon the light cones have changed in such a way that we have a ''black hole'' type of solution. A second feature is that there are no curvature singularities in the region between these two horizons, which would have been an interesting example of an inhomogeneous singularity. A third feature is that the apparent singularities of the Gowdy model are only horizons. This feature is not general, as we will see below.

The no-hair theorems show that only the Kerr family of metrics can represent true black holes. The other members of the enormous zoo of black hole-like solutions are untenable as black hole models because they contain curvature singularities on or outside of the outer horizon. This fact does not affect the Gowdy interiors, but instead makes them more interesting models. Another point about the ''black hole'' solutions is that many of them have been found as solutions of the Ernst equation. The simple fact that we have passed inside the horizon does not make the Ernst equation invalid, and one might expect that cosmological models with two commuting Killing vectors could be generated using the same techniques that were useful in the stationary axisymmetric case. This possibility will be discussed in detail in a forthcoming paper by the same authors. Recently, it was shown that particular Gowdy models can be generated from the data given on a specific hypersurface by applying solution generating techniques [12]. In the present article we will only make mention of some points related to this concept. As we mentioned above, the entire zoo of axisymmetric solutions with horizons could be laboriously converted to Gowdy models and their features studied. In this paper we only plan to give a few of the more interesting solutions. These solutions can be broken down into three categories, each one with a representative metric or class of metrics. These three are:

(1) Simple solutions—the Zipoy-Voorhees metric,
(2) More complicated Kerr-like solutions—Tomimatsu-Sato metrics,
(3) Complicated curvature-singularity-horizon behavior on the ''horizons''—the Erez-Rosen metric.

The models we will present correspond to each of these three cases. The motivation for studying these classes of solutions arises from our previous analysis of the Kerr metric inside the horizons [11]. There it was shown that the section of the Kerr spacetime contained between the inner and outer horizons can be reinterpreted as an $S^1 \times S^2$ Gowdy cosmological model which, in the terminology of Isenberg and Moncrief [13], corresponds to a nongeneric model. This implies that the curvature is bounded along paths which approach the Big Bang and Big Crunch singularities that correspond to the inner and outer horizon of the Kerr metric. The regular behavior of the curvature at a cosmological singularity indicates that the latter could become a Cauchy horizon and the spacetime could be extended beyond the singularity to include nonglobally hyperbolic acausal regions. In turn, this would imply a violation of the strong cosmic censorship conjecture. Accordingly, the cosmological sector of the Kerr spacetime can be considered as a counterexample of the strong cosmic censorship. The question arises whether there exist more general counterexamples. Since generic cosmological Gowdy models [13] are characterized by cosmological singularities with unbounded curvature, which excludes the possibility of extending the spacetime into acausal regions, the search for such counterexamples within the classes of Gowdy models, is equivalent to the search for nongeneric models. We will show that all the examples presented in this work basically belong to the class of generic models. This supports the conclusion of Ref. [13] that only a very small set of Gowdy spacetimes can be extended into an acausal region, across a Cauchy horizon.

In the terminology of string theory, the examples presented here are time and angular dependent backgrounds, Gowdy models, which could be of importance in the context of S-branes. It is not the purpose of this work to analyze the possibility of defining the corresponding models as i-models as has been done with iD-branes in [5] and for a simple $4D$ model in [6], and recently for the Kerr metric in [14,15]. A complete analysis of generalized S-brane solutions corresponding to the three categories of metrics considered here is in progress and will be reported on elsewhere. In particular, we have analyzed a simple case of the Zipoy-Voorhees spacetime and found that its cosmological sector admits an analytical continuation. The resulting metric can be interpreted as describing the simplest regular S-brane solution [16]. This result solves the singularity problem of S-branes in string theory, without requiring the existence of additional parameters which imply a twist in space. This simple example seems to indicate that, after an appropriate analytical continuation, all the cosmological Gowdy solutions presented in this work are potential candidates for describing different physical configurations of singular and regular S-brane solutions. In general we expect these S-brane backgrounds to be gravitational fields rather than simple counterparts of homogeneous cosmologies.

Gowdy models are the simplest example of a true field theory. This has always been one of the major uses of Gowdy models, that is, as a gravitational field theory that (in the polarized case) has simple exact solutions. Gowdy cosmologies have been used intensively as toy models for studying the nature of cosmological singularities. These studies have contributed to understand the nature of the singularities that form during a gravitational collapse, a long standing problem which now seems to be solved in quite general terms [17]. In the special case of Gowdy cosmologies, several numerical analysis have been performed to show that almost all of these models become asymptotically velocity term dominated (AVTD) near the singularities [13]. In simple terms, this behavior implies that near the singularity each point in space is characterized by a different spatially homogeneous cosmology. We will show in this work that the cosmological sectors of all the metrics mentioned above belong to the class of AVTD spacetimes.

In Section II we present some general considerations for the transition from axisymmetric static, stationary solutions to $S^1 \times S^2$ time and angular dependent Gowdy models in the context of the horizon method. In sections III, IV, and V, we will present the models corresponding to each of these metrics mentioned above, and will analyze the behavior of the relevant metric functions, especially near the singularities and horizons. In Section VI we use the Ernst potential of the corresponding metrics to show in a simple manner that all the models to be presented here are characterized by an AVTD behavior near the cosmological singularities. Section VII is devoted to conclusions and suggestions for further research.

II. GENERAL CONSIDERATIONS

Axisymmetric solutions are often given in Lewis-Papapetrou form. Here we will use prolate spheroidal coordinates where the metric takes the form

$$ds^2 = -f(dt - \omega d\varphi)^2 + f^{-1}(x^2 - 1)(1 - y^2)d\varphi^2 + f^{-1}e^{2\gamma}(x^2 - y^2)\left[\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2}\right], \tag{3}$$

where f, ω, and γ are functions of x and y. One sometimes writes $f = A/B$, where A and B are also functions of x and y.

The variable x is the radial coordinate, and the outer and inner horizons (the reason for the quotation marks will become obvious later) are at $x = \pm 1$. For "black hole" solutions one takes $x > 1$. The quantity y is the angular variable, and it is usual to make the transformation $y = \cos\theta$, where θ is the ordinary polar angle. The cosmological sector of these metrics is the region where $-1 < x < +1$. In order to make contact with previous Gowdy formulations, we will define $x = \cos(e^{-\tau})$ in the cosmological region. In this region the term $dx^2/(x^2 - 1)$ changes sign, which allows us to interpret τ as a time coordinate. With these changes, the metric (3) between its horizons takes the form for unpolarized $S^1 \times S^2$ Gowdy models that has been used by several authors [13,18],

$$ds^2 = e^{-\lambda/2}e^{\tau/2}(-e^{-2\tau}d\tau^2 + d\theta^2) + \sin(e^{-\tau})[e^P d\chi^2 + 2e^P Q d\chi d\varphi + (e^P Q^2 + e^{-P}\sin^2\theta)d\varphi^2], \tag{4}$$

where χ is the t of (3) and $\partial/\partial\chi$ is now supposed to be a spacelike direction, and we have compactified in the χ direction by supposing that $0 \leq \chi \leq 2\pi$, with zero and 2π identified. The functions in this metric can be identified with A, B, γ and ω of (3) by

$$e^{-\lambda/2}e^{\tau/2} = \frac{B}{A}e^{2\gamma}(\cos^2 e^{-\tau} - \cos^2\theta) \tag{5}$$

$$e^P = -\frac{A}{B\sin e^{-\tau}}, \qquad Q = -\omega. \tag{6}$$

The form of the $d\varphi^2$ term is due to the fact that the two-metric in $d\chi$ and $d\varphi$ must have determinant $\sin^2\theta\sin^2(e^{-\tau})$ for the metric (4) to be a solution of the Einstein equations, a condition that (4) obeys.

There are a number of points that we will see in all of the three cases discussed below. The most important point is the sign of e^P in (4). In order to have the correct signature in (3), it is assumed that for $x > 1, A/B$ is positive, and for P to be real in (4), either A or B must change sign at $x = 1$. While this is the case for some of the metrics we will study, it is not true for all of them.

However, the equations for P and Q,

$$P_{,\tau\tau} - e^{-2\tau}\frac{(\sin\theta P_{,\theta})_{,\theta}}{\sin\theta} - e^{-2\tau} - \frac{e^{2P}}{\sin^2\theta}\times [(Q_{,\tau})^2 - e^{-2\tau}(Q_{,\theta})^2] - [e^{-\tau}\cot(e^{-\tau}) - 1]P_{,\tau} = 0, \quad (7)$$

$$Q_{,\tau\tau} - e^{-2\tau}Q_{,\theta\theta} - e^{-2\tau}\cot\theta Q_{,\theta} + 2(P_{,\tau}Q_{,\tau} - e^{-2\tau}P_{,\theta}Q_{,\theta}) - [e^{-\tau}\cot(e^{-\tau}) - 1]Q_{,\tau} = 0. \quad (8)$$

depend on P only through e^{2P} and derivatives of P which are invariant under the addition of $i\pi$ to P, so if A/B does not change sign, we can add $i\pi$ to P and take $e^P = |A/B|/\sin(e^{-\tau})$ (the variable τ runs from $-\ln\pi$ to $+\infty$, so $\sin(e^{-\tau})$ is always positive). Since the equation for Q is invariant under a change of sign of Q, we may take $Q = \pm\omega$ as we wish. Since the equations for λ (see Ref. [13])

$$\cot(e^{-\tau})\lambda_{,\theta} - 2e^{\tau}\left(P_{,\tau}P_{,\theta} + e^{2P}\frac{Q_{,\tau}Q_{,\theta}}{\sin^2\theta}\right) + \cot\theta[-e^{\tau}\lambda_{,\tau} + 2e^{\tau}P_{,\tau} + e^{\tau} + 2\cot(e^{-\tau})] = 0, \quad (9)$$

$$\cot(e^{-\tau})(\lambda_{,\tau} - 1) - e^{\tau}[(P_{,\tau})^2 + e^{-2\tau}(P_{,\theta})^2] - e^{\tau}\frac{e^{2P}}{\sin^2\theta}[(Q_{,\tau})^2 + e^{-2\tau}(Q_{,\theta})^2] + e^{-\tau}[\cot^2(e^{-\tau}) + 4] + e^{-\tau}(-\cot\theta\lambda_{,\theta} + 2\cot\theta P_{,\theta}) = 0. \quad (10)$$

only depend on λ through its derivatives, we may also take

$$e^{-\lambda/2}e^{\tau/2} = |\frac{B}{A}e^{2\gamma}||\cos^2(e^{-\tau}) - \cos^2\theta|. \quad (11)$$

For all of the metrics we will investigate, $e^{2\gamma}$ is proportional to A, and B is positive definite (except at $x = \pm 1$ and $\theta = 0, \pi$, where the usual coordinate singularity of polar coordinates occurs), so the absolute value of $(B/A)e^{2\gamma}$ is not needed.

A second problem with the τ and θ coordinates is that there might be at least a coordinate singularity at $\pm\cos(e^{-\tau}) = \cos\theta$. Of course, this depends on the form of $e^{2\gamma}A/B$. In all of the cases we will consider, $e^{2\gamma}$ is proportional to a power of $\cos^2(e^{-\tau}) - \cos^2\theta$, while A/B times the rest of $e^{2\gamma}$ is regular and nonzero in the cosmological region. In most cases the power of $[\cos^2(e^{-\tau}) - \cos^2\theta]$ in (11) (Kerr and Erez-Rosen are the only exceptions) is not equal to zero, so $e^{-\lambda/2}$ is singular on a surface in the cosmological region. It can be shown by an explicit coordinate transformation given in the Appendix that for any power of $[\cos^2(e^{-\tau}) - \cos^2\theta]$, this singularity is only a coordinate effect. In the next sections we will study explicit examples for the application of this procedure.

III. THE ZIPOY-VOORHEES METRIC

The Zipoy-Voorhees [19] metric in Lewis-Papapetrou form and prolate spheroidal coordinates has the form of (3) with

$$\omega = 0, \qquad f = \left(\frac{x-1}{x+1}\right)^{\delta}, \qquad e^{2\gamma} = \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}, \quad (12)$$

where the constant parameter δ lies in the range $-\infty < \delta < +\infty$ with no other restrictions, which implies that we can take

$$A = (x^2 - 1)^{\delta}, \qquad B = (x+1)^{2\delta}. \quad (13)$$

Between the inner and outer horizons we take $x = \cos(e^{-\tau})$, and we have

$$A = (-1)^{\delta}\sin^{2\delta}(e^{-\tau}), \qquad B = (1 + \cos e^{-\tau})^{2\delta}. \quad (14)$$

Since in this "polarized" case ($Q = 0$) the equation for P only depends on derivatives of P, and we can add $-i\pi\delta$ to P and still have a solution, so we find that

$$e^P = \frac{\sin^{2\delta-1}(e^{-\tau})}{(1 + \cos e^{-\tau})^{2\delta}} \quad (15)$$

for any δ is a Gowdy solution, which can be seen by substituting this expression into (4) with $Q = 0$. In fact, ([15]) is the general solution (up to a trivial multiplicative constant) of (4) for P independent of θ, as can be shown by quadratures. This solution can also be written as $P = -2\delta Q_0(\cos e^{-\tau})/\sin e^{-\tau}$, where Q_0 is a Legendre function of the second kind, a form well-known in the $x > 1$ region. The expression for λ in this case is given by

$$e^{-\lambda/2}e^{\tau/2} = \frac{(1 + \cos e^{-\tau})^{2\delta}}{(\sin^2 e^{-\tau})^{\delta-\delta^2}}|\cos^2(e^{-\tau}) - \cos^2\theta|^{1-\delta^2}. \quad (16)$$

The expression multiplying the power of $|\cos^2 e^{-\tau} - \cos^2\theta|$ is analytic and nonzero between the two horizons and the singularity where $|\cos^2 e^{-\tau} - \cos^2\theta| = 0$ may be removed by the coordinate transformation of the Appendix. This solution is the simplest example of a Gowdy metric obtainable from an axisymmetric solution.

The curvature singularities of this spacetime can be found by analyzing the Kretschmann scalar, $K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, which in this case can be written as

$$K = 16\delta^2\left(\frac{\cos e^{-\tau} - 1}{\cos e^{-\tau} + 1}\right)^{2\delta}\frac{(\cos^2 e^{-\tau} - \cos^2\theta)^{2\delta^2-3}}{(\cos^2 e^{-\tau} - 1)^{2\delta^2+2}}L(\tau, \theta), \quad (17)$$

with

$$L(\tau,\theta) = 3(\cos e^{-\tau} - \delta)^2(\cos^2 e^{-\tau} - \cos^2\theta) + (\delta^2 - 1)\sin^2\theta \times[\delta^2 - 1 + 3\cos e^{-\tau}(\cos e^{-\tau} - \delta)]. \quad (18)$$

In the special case $\delta = 1$, in which we recover the Kantowski-Sachs model, there exists only one singularity at $\tau = -\ln\pi$. It is interesting to mention that for $\delta = -1$, the only singularity is situated at $\tau \to \infty$. Whereas in the Kantowski-Sachs model the singularity at $\tau = -\ln\pi$ corresponds to a Big Bang at the origin from which the universe expands forever free of singularities, in the dual case ($\delta = -1$) the universe possesses a regular origin and evolves asymptotically ($\tau \to \infty$) into a Big Crunch curvature-singularity.

Consider now the case $\delta \neq 1$. From Eq. (17) we can see that there is a singularity at $\cos e^{-\tau} = 1$ for $2\delta^2 - 2\delta + 2 > 0$ and at $\cos e^{-\tau} = -1$ for $2\delta^2 + 2\delta + 2 > 0$. Consequently, there exist true singularities at $\cos e^{-\tau} = \pm 1$ for any real values of δ. The apparent singularity at $\cos e^{-\tau} = \pm\cos\theta$, ($\delta^2 > 3/2$), can be removed by means of the coordinate transformation of the Appendix for any values of the angle θ with $\cos\theta \neq \pm 1$. When $\cos\theta = \pm 1$ we return to the latter case.

We have shown that the outer and inner horizons are actually surfaces of infinite curvature (naked singularities), and they make this metric untenable as a black hole model because it violates cosmic censorship, but between the horizons it represents a perfectly viable time and angular dependent background. In the terminology of Ref. [13], this solution belongs to the class of generic Gowdy models, i.e., cosmological models with unbounded curvature along paths which approach the singularity. The spacetime cannot be extended beyond the singularity to include acausal regions. Consequently, the cosmological sector of the Zipoy-Voorhees metric represents a globally hyperbolic manifold where the strong cosmic censorship conjecture holds.

IV. THE TOMIMATSU-SATO METRICS

The Tomimatsu-Sato metrics [20] are an infinite family of metrics with a parameter similar to the δ of the Zipoy-Voorhees metrics. In this case, as we will see, these models give "unpolarized" ($Q \neq 0$) Gowdy models between their horizons. For unpolarized Gowdy models Eqs. (7) and (8) contain e^{2P} as well as derivatives of P, so if we want (as we will for some of the metrics) to change P by adding $-i\pi\delta$ as we did for the Zipoy-Voorhees metrics, then there is an additional restriction from the fact that the equations are only invariant if $2i\pi\delta$ is an integer multiple of $2i\pi$, that is, δ an integer. The Tomimatsu-Sato solutions all have the equivalent of δ an integer, so we can always use them to create backgrounds depending on an angular variable and time.

In the Lewis-Papapetrou form, all these metrics have

$$A = A(x, y), \qquad B = B(x, y), \quad (19)$$

where A and B are polynomials in x and y. We also have

$$\omega = 2\delta\frac{q}{p}(1 - y^2)\frac{C}{A}, \quad (20)$$

where C is a polynomial in x and y, and p and q are numerical constants that obey $p^2 + q^2 = 1$. Finally, we have

$$e^{2\gamma} = \frac{1}{p^{2\delta}(x^2 - y^2)^{\delta^2}}A. \quad (21)$$

Yamazaki and Hori [21] have given expressions for A, B, C for the entire infinite family. In the original article of Tomimatsu and Sato A, B and C were given for $\delta = 1, 2, 3, 4$. Since the maximum powers of the polynomials A and B are $2\delta^2$, and all of the powers of x and y exist, the polynomials quickly become very cumbersome, so we will not try to give even all of the four models given by Tomimatsu and Sato. For $\delta = 1$ we have

$$A = p^2(x^2 - 1) - q^2(1 - y^2), \quad (22)$$

$$B = (px + 1)^2 + q^2y^2, \quad (23)$$

$$C = -px - 1. \quad (24)$$

This is just the Kerr metric already studied in Ref. [11]. The only model we will study in detail is the $\delta = 2$ model, where

$$A = [p^2(1 - x^2)^2 + q^2(1 - y^2)^2]^2 + 4p^2q^2(1 - x^2)(1 - y^2)(x^2 - y^2)^2, \quad (25)$$

$$B = \{p^2(1 + x^2)(1 - x^2) + q^2(1 + y^2)(1 - y^2) + 2px(1 - x^2)\}^2 + 4q^2y^2\{px(1 - x^2) - (px + 1)(1 - y^2)\}^2, \quad (26)$$

$$C = p^3x(1 - x^2)\{-2(1 + x^2)(1 - x^2) + (x^3 + 3)(1 - y^2)\} + p^2(1 - x^2)\{-4x^2(1 - x^2) + (3x^2 + 1)(1 - y^2)\} + q^2(px + 1)(1 - y^2)^3. \quad (27)$$

For $x = \cos(e^{-\tau})$, $y = \cos\theta$, we have

$$A = p^4\sin^8 e^{-\tau} + q^4\sin^8\theta + 2p^2q^2\sin^2 e^{-\tau}\sin^2\theta \times\{2\sin^4 e^{-\tau} + 2\sin^4\theta - 3\sin^2 e^{-\tau}\sin^2\theta\} \quad (28)$$

$$B = \{p^2\sin^2 e^{-\tau}(1 + \cos^2 e^{-\tau}) + q^2\sin^2\theta(1 + \cos^2\theta) + 2p\cos e^{-\tau}\sin^2 e^{-\tau}\}^2 + 4q^2\cos^2\theta \times\{p\cos e^{-\tau}\sin^2 e^{-\tau} - (p\cos e^{-\tau} + 1)\sin^2\theta\}^2, \quad (29)$$

$$
\begin{aligned}
C = {} & p^3 \cos e^{-\tau} \sin^2 e^{-\tau} \{-2(1+\cos^2 e^{-\tau})\sin^2 e^{-\tau} \\
& + (3+\cos^3 e^{-\tau})\sin^2\theta\} + p^2 \sin^2 e^{-\tau} \\
& \times \{-4\cos^2 e^{-\tau} \sin^2 e^{-\tau} + (3\cos^2 e^{-\tau}+1)\sin^2\theta\} \\
& + q^2(p\cos e^{-\tau}+1)\sin^6\theta. \qquad (30)
\end{aligned}
$$

These expressions can now be used in (5) and (6) to give P, Q and $e^{-\lambda/2}e^{\tau/2}$. In Figures 1 and 2 we give the behavior of $P - m = \ln(\frac{|A|}{|B|})$ [i.e. $m = -\ln(\sin e^{-\tau})$] and $Q = 2(q/p)\sin^2\theta(C/A)$ as functions of θ for several values of τ.

In Ref. [11] the behavior of P was that of a single peak in P as a function of θ that varied in width and height as τ varied from $-\ln\pi$ to $+\infty$. In the $\delta = 2$ case we can see from Figs. 1(a)–1(g) that there are two new peaks that move in θ as τ advances (near $\tau = -\ln\pi$ and as $\tau \to \infty$ the peaks are too small to appear in the figures). Figs. 2(a)–2(g) show the behavior of Q as a function of θ for the same values of τ as in Fig. 1. The same two new peaks appear in Q. The flat lines at $\tau = -\ln\pi$ and $\tau = \infty$ can be seen as square peaks where the extra peaks have been "squashed" into the vertical axes at $\theta = 0$ and $\theta = \pi$. Since the Tomimatsu-Sato polynomials A and B have powers up to $2\delta^2$, we can expect similar behavior for higher δ models, with a rapidly growing number of peaks as δ grows. Since the amplitude of $P - m$ varies considerably, we have given $P - m$ for $\theta = 0$ as a function of τ in Fig. 3.

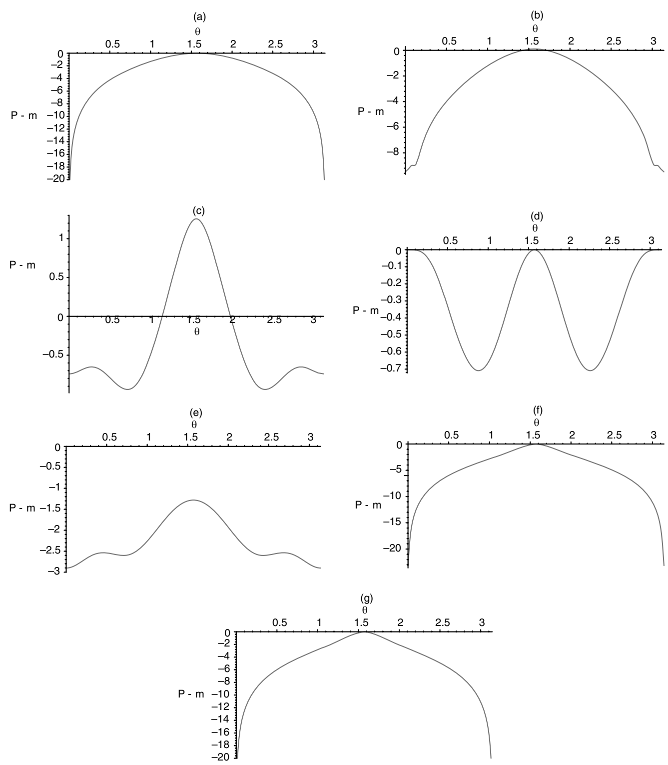

FIG. 1. The evolution in τ of $P - m$, [$m = -\ln(\sin e^{-\tau})$], as a function of θ. (a) $\tau = -\ln\pi$, (b) $\tau = -1.1$, (c) $\tau = -0.75$, (d) $\tau = -\ln(\pi/2)$, (e) $\tau = 0$, (f) $\tau = +5$, (g) $\tau = +\infty$.

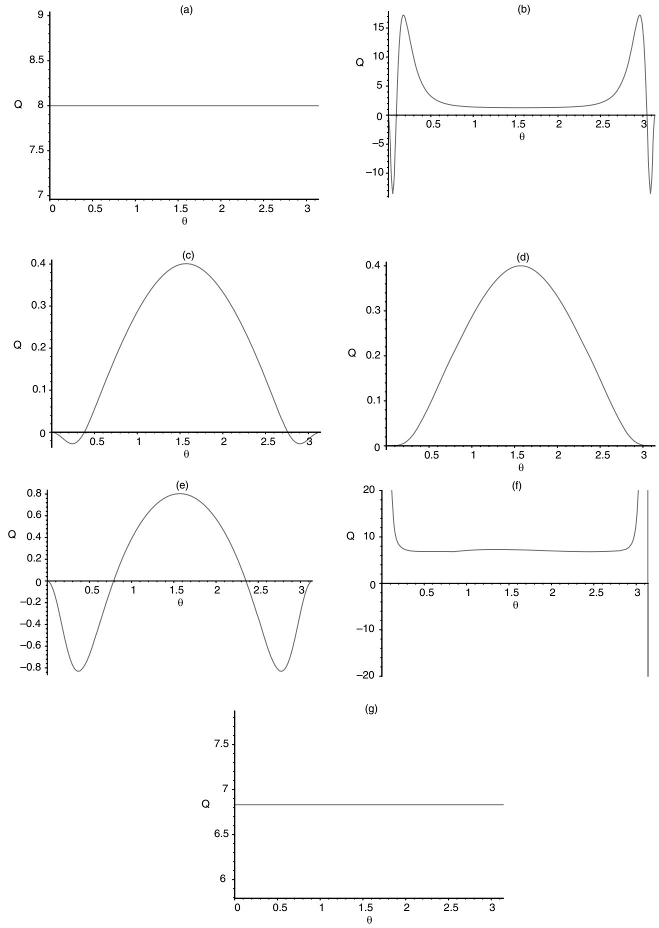

FIG. 2. The evolution in τ of Q as a function of θ. (a) $\tau = -\ln\pi$, (b) $\tau = -1.1$, (c) $\tau = -0.75$, (d) $\tau = -\ln(\pi/2)$, (e) $\tau = 0$, (f) $\tau = +5$, (g) $\tau = +\infty$.

There are a number of points about A and B. For $\delta = 1$, $e^{\lambda/2}e^{\tau/2}$ has no coordinate singularity at $\cos^2 e^{-\tau} = \cos^2\theta$, since $e^{2\gamma}$ has a factor of $[\cos^2 e^{-\tau} - \cos^2\theta]^{-1}$ that cancels the same factor in Eq. (11). For all other values of δ there is a coordinate singularity where $\cos^2 e^{-\tau} = \cos^2\theta$. Another problem is the possible existence of singularities (which can be either coordinate singularities or curvature singularities) corresponding to points where A or B are zero.

We can check whether either of these polynomials is zero. If we define $V = \sin^2 e^{-\tau}$ and $W = \sin^2\theta$, we find that

$$
A = (p^2V^2 + q^2W^2)^2 + 4p^2q^2VW(W - V)^2. \qquad (31)
$$

Notice that this expression is the sum of two positive terms ($V, W \geq 0$) so the second term in A can only be zero if W or V is zero (the first term being zero then implies $W = V = 0$) or if $W = V$ and $(p^2 + q^2)V^2 = V^2 = 0$. Both of these conditions imply that the only zeros of A

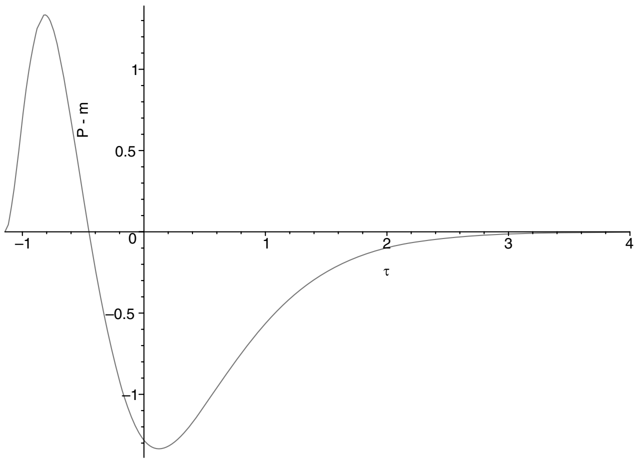

FIG. 3. $P - m$ at $\theta = 0$ versus τ.

occur at $\cos e^{-\tau} = \pm 1$. Tomimatsu and Sato [20] and Hikida and Kodama [22] have shown that these singularities are only the inner and outer horizons of the model.

The other possibility of singular points is where $B = 0$. From (29) we can see that B is also the sum of two positive terms, and these terms must both be zero. There are various possibilities. The second term is zero if $\cos\theta = 0$ or $q = 0$ ($p = 1$) or

$$p \cos e^{-\tau} \sin^2 e^{-\tau} - (p \cos e^{-\tau} + 1) \sin^2\theta = 0 \qquad (32)$$

For $\cos\theta = 0$, the first term in B reduces to (using $\sin 2e^{-\tau} = 1 - \cos^2 e^{-\tau}$ and $p^2 + q^2 = 1$)

$$p^2 \cos^4 e^{-\tau} + 2p \cos^3 e^{-\tau} - 2p \cos e^{-\tau} - 1 = 0. \qquad (33)$$

The four solutions of this equation are real only for $|p| > 1$ (impossible) except for two cases, $p = +1$, $\cos e^{-\tau} = +1$, $p = -1$, $\cos e^{-\tau} = 1$. These are "ring" singularities on the horizons of the Gowdy models. This is the limit of the well-known ring singularity of the Tomimatsu-Sato $\delta = 2$ metrics that is always outside the Gowdy region except for the case we have given. This singularity is a curvature-singularity [20,22].

The next possibility is $q = 0$ ($p = \pm 1$) where the first term in B is zero if $\sin e^{-\tau} = \pm 1$ or if $\cos^2 e^{-\tau} + 2\cos e^{-\tau} + 1 = 0$ (which has a real solution only for $\cos e^{-\tau} = -1$). These are the horizons of the Tomimatsu-Sato metric, and the curvature invariants are finite except for the ring singularity when $p = \pm 1$.

Finally, we can take any solution where (32) is satisfied. Solving for $\sin 2\theta$ and plugging the result into the first term of B, we find a fifth order expression in $\cos e^{-\tau}$ that must be zero. Plotting this expression as a function of $e^{-\tau}$ for p between 0 and 1, we find that it is never zero except for $p = \pm 1$ for $\cos e^{-\tau} = +1$. Each of these gives $\sin^2\theta = 0$, so the singularity is just the ordinary coordinate singularity of spherical coordinates at the poles.

We have seen that in the general case $q \neq 0$ the cosmological sector of this solution is contained inside the inner and outer horizons which are hypersurfaces of finite curvature. This implies that the horizons could become Cauchy horizons and the spacetime could be extended beyond them. In this sense, the corresponding Gowdy model is nongeneric. However, a true ring singularity exists outside the outer horizon which prevents the spacetime from being extended into further acausal regions and indicates the existence of a generic Gowdy model. Thus, the cosmological sector of the Tomimatsu-Sato metric with $\delta = 2$ could be interpreted as a "mixed" model in which the strong cosmic censorship conjecture can be violated along the horizons (nongeneric behavior), except along the ring singularity (generic behavior). It would be interesting to analyze in detail this peculiar behavior. We intend to attack this task in a forthcoming work.

V. THE EREZ-ROSEN SOLUTION

The Erez-Rosen solution [23] belongs to the class of static solutions and its metrics functions read (q is an arbitrary constant)

$$\omega = 0, \qquad (34)$$

$$f = \exp(2\psi), \quad \psi = \frac{1}{2} \ln\frac{x-1}{x+1} + q\tilde{\psi}, \qquad (35)$$

$$\gamma = \frac{1}{2} \ln\frac{x^2-1}{x^2-y^2} + q\tilde{\gamma}, \qquad (36)$$

where

$$\tilde{\psi} = \frac{1}{2}(3y^2 - 1)\left[\frac{1}{4}(3x^2 - 1)\ln\frac{x-1}{x+1} + \frac{3}{2}x\right], \qquad (37)$$

and

$$\tilde{\gamma} = \left(1 + \frac{1}{2}q\right)\ln\frac{x^2-1}{x^2-y^2} - \frac{3}{2}(1-y^2)\left(x\ln\frac{x-1}{x+1} + 2\right) + \frac{9}{16}q(1-y^2)\left[x^2 + 4y^2 - 9x^2y^2 - 4/3 + x(x^2 + 7y^2 - 9x^2y^2 - 5/3)\ln\frac{x-1}{x+1} + \frac{1}{4}(x^2-1)(x^2+y^2-9x^2y^2-1)\ln^2\frac{x-1}{x+1}\right]. \qquad (38)$$

The Erez-Rosen solution is interpreted as describing the exterior gravitational field of a nonspherically symmetric mass distribution with quadrupole moment proportional to the arbitrary constant q. When the mass quadrupole moment vanishes we recover the standard Schwarzschild spacetime with $x = r/M - 1$ and $y = \cos\theta$. In general, for the metric functions of the Erez-Rosen solution to be well-defined one has to demand that $|x| > 1$ (and $|y| \leq 1$), although the limiting value $x = \pm 1$

can also be included by using an appropriate procedure [24]. For the purposes of the present work, however, we will include that limiting value in the time-dependent sector, which now corresponds to $-1 \leq x \leq 1$. In this case, there is no problem with the argument of the logarithmic function appearing in the metric functions (35) and (36). In fact, one can show that the argument $(x-1)/(x+1)$ can be replaced everywhere by its absolute value and the resulting expressions remain valid as exact solutions of the vacuum field equations.

In order to identify this solution as an $S^1 \times S^2$ Gowdy model in the region $-1 \leq x \leq 1$, we take, as before, $y = \cos\theta$ and $x = \cos e^{-\tau}$. The metric functions A and B can then be chosen as

$$A = -\sin^2 e^{-\tau} \exp(2q\tilde{\psi}) \quad \text{and} \quad B = (1 + \cos e^{-\tau})^2 \tag{39}$$

As in the previous cases, the function γ turns out to be proportional to A since $\exp(2\gamma) = A(\cos^2 e^{-\tau} - \cos^2\theta)^{-1}\exp(2q\tilde{\gamma} - 2q\tilde{\psi})$. Consequently, the metric functions for the corresponding Gowdy model can be written as

$$Q = 0 \tag{40}$$

$$e^P = \frac{\sin e^{-\tau}}{(1 + \cos e^{-\tau})^2} e^{2q\tilde{\psi}}, \tag{41}$$

$$e^{-\lambda/2+\tau/2} = (1 + \cos e^{-\tau})^2 e^{2q\tilde{\gamma} - 2q\tilde{\psi}}. \tag{42}$$

We now have a time and angular dependent background which evolves in time from $\tau = -\ln\pi$ to $\tau \to \infty$.

Since $Q = 0$ this is a polarized Gowdy model, so there is a separable general solution for P as a sum over Legendre polynomials,

$$P = \sum_{l=0}^{\infty} T_l(e^{-\tau}) P_l(\cos\theta). \tag{43}$$

Erez-Rosen is a particular solution where the expression (41), using (37), reduces to

$$P = -2Q_0(\cos e^{-\tau})P_0(\cos\theta) - 2qQ_2(\cos e^{-\tau})P_2(\cos\theta), \tag{44}$$

where the Q_l are Legendre functions of the second kind. This is a relatively simple function, and if we use $P_0 = 1$ to write

$$P + 2Q_0(\cos e^{-\tau}) = -2qQ_2(\cos e^{-\tau})P_2(\cos\theta), \tag{45}$$

$P + 2Q_0$ is simply a time-dependent amplitude times $P_2(\cos\theta)$, a very well-known function. For completeness, however, we will graph $P_2(\cos\theta)$ versus θ in Figure 4 and $Q_2(\cos e^{-\tau})$ ($1/q$ times the amplitude of $P + 2Q_0$ at $\theta = 0$) versus τ in Fig. 5. Notice that $Q_2(\cos e^{-\tau})$ is just $Q_2(\cos z)$ "stretched" by the exponential factor $e^{-\tau}$.

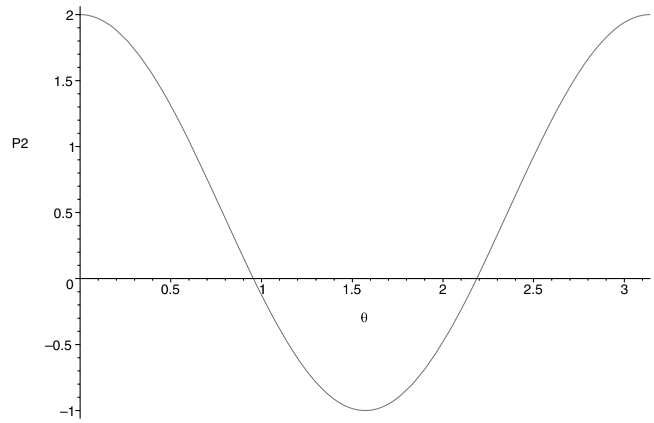

FIG. 4. The Legendre polynomial $P_2(\cos\theta)$ versus θ.

One can calculate the Kretschmann scalar corresponding to this spacetime and show that no curvature singularities exist in the region $-\ln\pi < \tau < \infty$. However, the limiting values of this region show interesting behavior. To see this, let us turn back to the Erez-Rosen metric since we know that these limiting values of the cosmological evolution correspond to the horizon of the exterior "black hole" solution. The horizon in the Erez-Rosen metric is determined by the zeros of the norm of the Killing vector $\xi^\mu = (\partial_t)^\mu$ associated with the time coordinate t:

$$\xi^\mu \xi_\mu = \frac{\exp[3qxP_2(y)]}{(x+1)^{1+qP_2(x)P_2(y)}}(x-1)^{1+qP_2(x)P_2(y)}. \tag{46}$$

Here the horizon turns out to be at $x = +1$ with the interesting feature that its existence depends on the values of the constant q and the angular coordinate y. In fact, the hypersurface $x = +1$ is a horizon only if the condition

$$1 + \frac{q}{2}(3y^2 - 1) > 0, \tag{47}$$

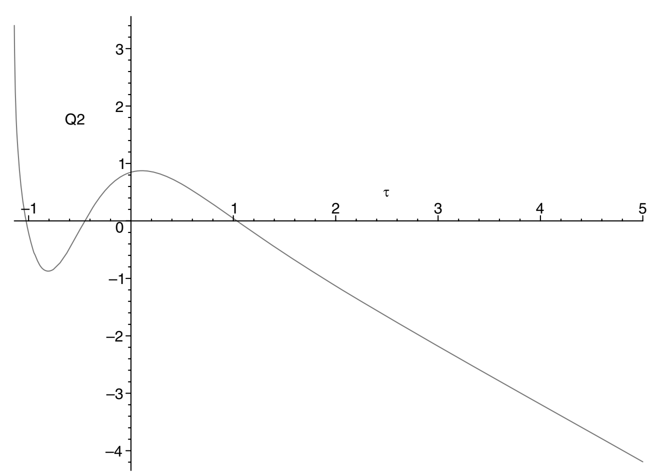

FIG. 5. The Legendre function $Q_2(\cos e^{-\tau})$ versus τ.

is satisfied, i.e., for

$$q > 0 \quad \text{and} \quad \frac{1}{3}(1 - 2/q) \leq y^2 \leq 1, \tag{48}$$

$$q < 0 \quad \text{and} \quad 0 \leq y^2 \leq \frac{1}{3}(1 - 2/q). \tag{49}$$

Notice that when these conditions are not satisfied the norm of the Killing vector diverges (Killing singularity). An analysis of these conditions leads to the following conclusions: For $-1 \leq q \leq 2$ the horizon occupies the entire hypersurface $x = 1$, i.e., it does not depend on the angular coordinate y. This, of course, includes the limiting case $q = 0$ (the Schwarzschild limit) in which no Killing singularity exists at $x = 1$. If we represent the hypersurface $x = 1$ by a circle, the horizon coincides with the entire circle. For $q < -1$, the horizon is symmetric with respect to the equatorial plane $y = 0$ ($\theta = \pi/2$), but it does not cover the entire hypersurface $x = 1$. Indeed, around the symmetry axis ($y = \pm 1$) a Killing singularity appears that extends from $\theta = 0$ to $\theta = \theta_- = \arccos\sqrt{(1 + 2/|q|)/3}$. The arc-length of the section occupied by the Killing singularity reaches its maximum value when $q \to -\infty$, i.e., for $\theta_{sing} = \arccos\sqrt{1/3}$. For positive values of q and $q > 2$, the horizon at $x = 1$ is symmetric with respect to the symmetry axis ($y = \pm 1$) and reaches its maximum arc-length with respect to the axis at $\theta_{\rm hor} = \pi/2 - \arccos(1/\sqrt{3})$ for $q \to \infty$. The remaining section of the hypersurface $x = 1$ is covered by the Killing singularity. There is a particular angular direction

$$\theta_- = \arccos\sqrt{\frac{1}{3}\left(1 + \frac{2}{|q|}\right)}, \quad \text{for} \quad q < -1, \tag{50}$$

or

$$\theta_+ = \arccos\sqrt{\frac{1}{3}\left(1 - \frac{2}{q}\right)}, \quad \text{for} \quad q > 2, \tag{51}$$

which determines the boundary on $x = 1$ that separates the horizon from the Killing singularity. In fact, due to the axial symmetry this boundary corresponds to a sphere S^1. So we see that in the case $q < -1$ and $q > 2$, the hypersurface $x = 1$ contains a horizon and a Killing singularity as well. The corresponding Gowdy cosmological model is defined "inside" this hypersurface.

Since the analytic expression for the Kretschmann scalar in this case is rather cumbersome, we only quote the results of our analysis. There exists a true curvature-singularity at $x = -1$, independent of the values of q and the angular variable y. (This corresponds to the Schwarzschild singularity at the origin of coordinates.) A second singularity is situated on the hypersurface $x = 1$ for all values of q and y, except on the symmetry axis ($\theta = 0$) and for the special direction θ_- (or θ_+), i.e., on the boundary between the "horizon" and the Killing singularity. In those particular directions the Kretschmann scalar remains constant. This means that the cosmological model inside the "singular horizon" evolves from a true Big Bang singularity at $x = -1$ ($\tau = -\ln\pi$) into a true Big Crunch singularity everywhere at $x = 1$ ($\tau \to \infty$), except in the special directions $\theta = 0, \theta_-, \theta_+$. From the point of view of the exterior ($|x| > 1$) "black hole" spacetime, the interior ($|x| \leq 1$) cosmological model can be reached only through the "angular windows" located at $\theta = 0, \theta_-, \theta_+$. The particular behavior at $\theta = 0$ can be associated with the coordinate singularity present in prolate spheroidal coordinates at the poles.

The resulting Gowdy cosmology is contained within the interval $-1 \leq x \leq 1$. At the hypersurface $x = -1$ the curvature blows up, indicating that the Gowdy model is generic and cannot be extended into acausal regions with $x < -1$. The outer horizon ($x = 1$), however, presents a different structure. The curvature is unbounded everywhere at $x = 1$ (a generic model), except at the "angular windows" (a nongeneric model at $\theta = \theta_-, \theta_+$). This could be interpreted as an example of a partly generic and partly nongeneric model. A more detailed analysis, probably associated with a numerical investigation of the limiting surface, would be necessary to clarify the validity of the strong cosmic censorship conjecture in this case.

VI. THE AVTD BEHAVIOR

To determine that a Gowdy model is AVTD near a singularity one basically has to prove that the following two conditions are satisfied: (i) the model is a solution of Einstein's vacuum field equations, and (ii) the model approaches in an appropriate manner the velocity term dominated (VTD) solution near the singularity. The VTD solution can be obtained by neglecting all the spatial derivatives in the field equations. In [11] we have proposed an alternative but equivalent way which is based on the following simple statement: A model is AVTD if it approaches a VTD solution near the singularity. In particular, it is well-known that the Kantowski-Sachs (KS) solution corresponds to an AVTD Gowdy model. So if we can show that a specific solution approaches the KS near the singularity, we also show that it is AVTD. To manipulate the solutions we have presented above we will use the corresponding Ernst potential that has the following properties: (i) it contains all the information about the solution because it can be used to calculate all the metric functions, and (ii) it is a scalar because in can be written in terms of the norm of the Killing vectors associated with the Gowdy spacetime. These two properties guarantee that our results are complete and invariant.

Let us recall that the main field Eqs. (7) and (8) are equivalent to the Ernst equation [11]

$$(1 - \xi\xi^*)[\nabla^2\xi + \nabla \ln(\sin T \sin\theta)\nabla\xi] + 2\xi^*(\nabla\xi)^2 = 0, \tag{52}$$

where ∇ represents a complex vector operator

$$\nabla = (\partial_T, i\partial_\theta), \qquad T = e^{-\tau}, \tag{53}$$

and the Ernst potential is defined as

$$\xi = \frac{1 - \sin e^{-\tau} e^P - iR}{1 + \sin e^{-\tau} e^P + iR},$$
$$R_{,T} = \frac{\sin e^{-\tau}}{\sin\theta} e^{2P} Q_{,\theta}, \qquad R_{,\theta} = \frac{\sin e^{-\tau}}{\sin\theta} e^{2P} Q_{,T}. \tag{54}$$

Here an asterisk represents complex conjugation. In the particular case of the KS solution, the Ernst potential turns out to be $\xi_{KS} = \cos e^{-\tau}$ (it corresponds to the case $\delta = 1$ of the Zipoy-Voorhees metric) so that near the singularities $\tau \to -\ln\pi$ and $\tau \to \infty$ the AVTD behavior is -1 and $+1$, respectively.

We now turn back to analyze the explicit examples presented in the previous Sections. Let us first consider the special polarized case, $Q = 0$; then $R = 0$ and the Ernst potential becomes real. From Eqs. ((19) and (41)) it is straightforward to calculate the Ernst potential for the Zipoy-Voorhees metric

$$\xi_{ZV} = \frac{(1 + \cos e^{-\tau})^\delta - (1 - \cos e^{-\tau})^\delta}{(1 + \cos e^{-\tau})^\delta + (1 - \cos e^{-\tau})^\delta}, \tag{55}$$

and for the Erez-Rosen metric

$$\xi_{ER} = \frac{(1 + \cos e^{-\tau}) - (1 - \cos e^{-\tau}) e^{2q\tilde{\psi}}}{(1 + \cos e^{-\tau}) + (1 - \cos e^{-\tau}) e^{2q\tilde{\psi}}}. \tag{56}$$

From these expressions it is clear that the limits of both solutions near the singularities $\tau \to -\ln\pi$ and $\tau \to \infty$ coincide with the limits of the KS solution.

In the more general case of the Tomimatsu-Sato metrics, we need to calculate explicitly the function R which enters the Ernst potential (54). This involves integrals of the functions A, B, and C given in Eqs. (28)–(30), respectively. The resulting expressions become rather cumbersome, but their analysis is straightforward and shows that the Tomimatsu-Sato metrics with $\delta = 1$ and $\delta = 2$ behave near the singularities as the Kantowski-Sachs metric. In fact, the case $\delta = 1$ corresponds to the Kerr-Gowdy metric presented in Ref. [11] where its AVTD behavior was also demonstrated by means of other different methods. The case $\delta = 2$ with $q = 0$ and $p = 1$ is easy to analyze since the corresponding Ernst potential

$$\xi_{TS2} = \frac{(1 + \cos e^{-\tau})^2 + (1 - \cos e^{-\tau})^2}{(1 + \cos e^{-\tau})^2 - (1 - \cos e^{-\tau})^2}, \tag{57}$$

turns out to be the inverse of the potential ξ_{ZV} with $\delta = 2$. Then, the behavior near the singularities is as in the KS metric.

The above analysis shows that all the models presented in this work are AVTD near the singularities.

VII. CONCLUSIONS

Time-dependent backgrounds can be generated in a systematic way by means of what was called "horizon methods" in [7]. In [11] we applied this procedure to the Kerr metric and were able to obtain a time and angular dependent background, a Gowdy $S^1 \times S^2$ model. In this work we have extended the previous analysis and have been able to exhibit backgrounds depending on time and an angular variable, Gowdy $S^1 \times S^2$ models that correspond to each of the three categories of axisymmetric solutions, the simple Zipoy-Voorhees metric, Tomimatsu-Sato metrics and the Erez-Rosen metric which has complicated curvature-singularity-horizon behavior.

In the case of the Zipoy-Voorhees metric we have shown that the outer and inner horizons are actually surfaces of infinite curvature (naked singularities) and the models are then untenable as black hole models because they violate cosmic censorship. However, between the horizons it represents a viable time and angular dependent background.

The Gowdy models generated by the Tomimatsu-Sato metrics we studied (including the Kerr metric of Ref. [11]) are unique among our three metrics in that they are unpolarized Gowdy models ($Q \neq 0$). This means that it is the only solution that cannot be written as a linear sum over eigenfunctions (Legendre polynomials). Another point is that the cosmological singularities are just horizons except for the extreme $\delta = 2$ model with $p = \pm 1$ and $q = 0$, where there is a ring curvature-singularity at $\theta = \pi/2$ on the horizons.

In the case of the Erez-Rosen metric we have seen that it can be interpreted as a polarized Gowdy model inside the "singular horizon". This horizon is special in the sense that for a large range of values of the "quadrupole moment" it allows the existence of a "regular horizon" (with an S^1 topology) through which the interior time and angular dependent sector can be reached.

By using the Ernst potential, which completely describes the cosmological sectors of the above metrics, we were able to demonstrate that all of them approach the KS solution near the singularities. On the other hand, it is known that the KS solution is AVTD near the singularities. From these two facts, we conclude that all the solutions considered in this work are AVTD near the Big Bang and Bing Crunch singularities.

Our analysis of the curvature behavior at the cosmological singularities present in all the models investigated in this work shows that all of them can be considered basically as generic Gowdy models. This is to say that they do not admit in general an extension beyond the singularities into nonglobally hyperbolic acausal regions, and that the strong cosmic censorship conjecture holds in all of these examples. Nevertheless, we have found a curious situation in the Erez-Rosen spacetime where the outer horizon contains a sphere S^1 with bounded curvature, indicating the presence of a nongeneric Gowdy model. In principle, one could imagine extending the corresponding nongeneric model across the Cauchy horizon situated on the regular sphere, into nonglobally acausal regions. This would indicate the presence of an "angular window" in the outer horizon of the Erez-Rosen cosmological sector which could be used to impose a violation of the strong cosmic censorship. This problem would probably imply the implementation of especially adapted numerical tools, a task which is beyond the scope of this work.

The Tomimatsu-Sato metrics and the Erez-Rosen metric represent long-wavelength unpolarized and polarized "gravitational waves" propagating around the universe (the Zipoy-Voorhees metric is a degenerate example that does not depend on the angular variable). The field theory S-brane solutions generated from these metrics may be of interest as simple field theory brane models.

S-brane solutions have been generated by means of the Kerr metric. These solutions have been proposed and studied independently in [14,15]. We have outlined the possibility of using the procedure presented here and in [11] to generate models that would correspond to generalized S-brane solutions. This possibility has been realized recently in Ref. [16] where we have shown that the cosmological sector of the Zipoy-Voorhees family of solutions can be used as starting metric to generate, via an analytical continuation procedure, S-brane regular solutions. In particular, we found that the limiting case of the Zipoy-Voorhees metric with $\delta = 1$ (i.e., the Schwarzschild spacetime) admits two different analytical continuations. The first one is performed "outside the horizon" and leads to the so called singular $S0$-brane solution of string theory. The second one, instead, is performed "inside the horizon" and generates a new $S0$-brane solution which does not contain any naked timelike singularity and therefore can be used to describe the process of formation and decay of an unstable D-brane. The time-dependent solutions corresponding to all the three different categories of axisymmetric solutions presented here also depend on an angular coordinate. An analysis of the models that one would obtain by introducing an i-factor in the coordinates is in progress and the general S-brane solutions obtained by applying this procedure will be discussed elsewhere.

ACKNOWLEDGMENTS

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APPENDIX

As we have mentioned several times, the only obvious singularity in the three Gowdy metrics we have studied is the one where $\cos^2 e^{-\tau} = \cos^2\theta$ ($x^2 = y^2$ in Lewis-Papapetrou coordinates) which makes $e^{-\lambda/2+\tau/2}$ either zero or infinity on that spacetime surface. One could check that curvature invariants are nonsingular on that surface, but for our metrics it is simple enough to find a coordinate transformation that makes the metric nonsingular.

In the $\tau\theta$ sector of the Gowdy metric we can write the two-dimensional metric, $d\sigma^2 = e^{-\lambda/2} e^{\tau/2}(-e^{-2\tau} d\tau^2 + d\theta^2)$, and for our metrics we have

$$d\sigma^2 = \frac{W(\tau, \theta)}{|\cos^2 e^{-\tau} - \cos^2\theta|^s}(-e^{-2\tau} d\tau^2 + d\theta^2), \quad \text{(A1)}$$

where the power s may be positive or negative, and $W(\tau, \theta)$ is nonsingular for $\tau \neq -\ln\pi$ or $\tau \neq \infty$.

For the Zipoy-Voorhees metric,

$$W(\tau, \theta) = \frac{(\sin^2 e^{-\tau})^{\delta^2+\delta}}{(\cos e^{-\tau} - 1)^{2\delta}} \quad \text{(A2)}$$

and $s = \delta^2 - 1$. Notice that s is negative for $-1 < \delta < +1$ and positive otherwise, and W is nonsingular and nonzero except at $\tau = -\ln\pi$ or $\tau = \infty$.

For any Tomimatsu-Sato metric,

$$W(\tau, \theta) = \frac{1}{p^{2\delta}} B(\tau, \theta), \quad \text{(A3)}$$

B a polynomial of order $2\delta^2$. The parameter s is $\delta^2 - 1$ and is positive for all $\delta > 1$. The function B is always of the form $(\alpha + \beta)(\alpha^* + \beta^*)$, α and β complex functions of τ and θ, so B is always positive and is zero only if $|\alpha + \beta| = 0$. For $\delta = 2$ we showed that this expression is zero only for special values of p and q given in Sec. III and only for $\tau = -\ln\pi$ and $\tau = \infty$. We can conjecture that this will also be true for any value of δ.

Finally, for the Erez-Rosen metric we have

$$W(\tau, \theta) = (1 + \cos e^{-\tau})^2 (\sin^2 e^{-\tau})^{2q+q^2} e^{2q\gamma^* - 2q\tilde{\psi}}, \quad \text{(A4)}$$

where

$$\gamma^* = -\frac{3}{2}\sin^2\theta\left[\cos e^{-\tau}\ln\left(\frac{1-\cos e^{-\tau}}{1+\cos e^{-\tau}}\right)+2\right]+\frac{9}{16}q\sin^2\theta \times\left[\cos^2 e^{-\tau}+4\cos^2\theta-9\cos^2 e^{-\tau}\cos^2\theta-\frac{4}{3} + \cos e^{-\tau}\left(\cos^2 e^{-\tau}+7\cos^2\theta-9\cos^2 e^{-\tau}\cos^2\theta-\frac{5}{3}\right) \times\ln\left(\frac{1-\cos e^{-\tau}}{1+\cos e^{-\tau}}\right)-\frac{1}{4}\sin^2 e^{-\tau}(-\sin^2 e^{-\tau} +\cos^2\theta-9\cos^2 e^{-\tau}\cos^2\theta)\ln^2\left(\frac{1-\cos e^{-\tau}}{1+\cos e^{-\tau}}\right)\right], \quad \text{(A5)}$$

and $s = 2q + q^2$. Notice that W is either zero or infinity only for $\cos^2 e^{-\tau} = \pm 1$, and so the only possible singularity not on the inner or outer horizons is where $\cos^2 e^{-\tau} = \cos^2\theta$. For $-2 < q < 0$, s is negative. For all other values of q it is positive.

If we now look at the general form of $d\sigma^2$ from (A1), and define $\xi = e^{-\tau}$, we have

$$d\sigma^2 = \frac{W(\xi,\theta)}{|\cos^2\xi - \cos^2\theta|^s}[-d\xi^2 + d\theta^2] \quad \text{(A6)}$$

$$= W(\xi,\theta)\left\{\frac{-d\xi^2 + d\theta^2}{|\sin(\xi+\theta)\sin(\theta-\xi)|^s}\right\}. \quad \text{(A7)}$$

Defining $w = \xi + \theta$ and $z = \theta - \xi$,

$$d\sigma^2 = W(w,z)\left[\frac{-dwdz}{|\sin w\sin z|^s}\right]. \quad \text{(A8)}$$

We can now define new coordinates u and v, where

$$du = \frac{dw}{|\sin w|^s}, \qquad dv = \frac{dz}{|\sin z|^s}, \quad \text{(A9)}$$

and by carefully checking constants of integration in the different regions where $\cos w$ and $\cos z$ have different signs, we can, in principle, find u and v as functions of w and z. In the uv coordinates we have

$$d\sigma^2 = -W(u,v)dudv, \quad \text{(A10)}$$

nonsingular as long as W is neither zero nor infinity.

The integrals necessary to solve (A9) are not tabulated for s noninteger, but if s is an integer, we can express u and v as finite sums of various powers of sines of ξ or θ and multiples of ξ and θ themselves for s negative, and as a finite series of sines divided by powers of cosines and logarithms of tangents for s positive. As an example, for the Tomimatsu-Sato $\delta = 2$ model given in Sec. IV (ignoring the absolute values),

$$du = \frac{dw}{\sin^3 w}, \qquad dv = \frac{dz}{\sin^3 z}, \quad \text{(A11)}$$

or

$$u = -\frac{1}{2}\frac{\cos(e^{-\tau}+\theta)}{\sin^2(e^{-\tau}+\theta)} + \frac{1}{2}\ln\left[\tan\left(\frac{e^{-\tau}}{2}+\frac{\theta}{2}\right)\right], \quad \text{(A12)}$$

$$v = -\frac{1}{2}\frac{\cos(\theta-e^{-\tau})}{\sin^2(\theta-e^{-\tau})} + \frac{1}{2}\ln\left[\left|\tan\left(\frac{\theta}{2}-\frac{e^{-\tau}}{2}\right)\right|\right]. \quad \text{(A13)}$$

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